summerschool UTRECHT

Summer School on Geometry

12-23 August 2024

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Course organisers. Dr. Álvaro del Pino Gómez and Wilfred de Graaf MSc.
Contact. w.f.degraaf@uu.nl
For more information and to apply: www.utrechtsummerschool.nl/courses/science/geometry.

Location. MIN2.02, 2th floor of the Minnaert building, Leuvenlaan 4, Utrecht.
It is easy to get by bicycle to the summer school location. On the website of Utrecht Summer School or in the pre-departure information, which becomes available after you have paid the course fee, you can find information about renting bicycles and using the Utrecht public transport system.

The Mathematical Institute will provide lunches on all ten days of the summer school, a barbecue on Monday 12 August and on Tuesday 20 August we will go out for a pancake.

Saturday/Sunday 10 and 11 August	**Key pick-up.** You will find the exact key pick-up location in the pre-departure information, which becomes available after you have paid the course fee.

Monday 12 August	**Prof. Dr. Gunther Cornelissen: Zeta functions** Zeta functions are generating functions with very strong analytic properties, whose special values and poles encode information from the number-theoretical, combinatorial, or geometrical input. The prototype is the Riemann zeta function, that can be used to study the distribution of prime numbers and properties of prime ideals, but there are also zeta functions that encode lengths of geodesics, walks in graphs, points on algebraic curves, spectra of operators, etc. The lecture will be a very gentle introduction to the theory, and in the second part, you pick your favourite zeta function and work with it yourself. **09.00** Welcome and registration **09.30** Lectures and exercises **12.30** Lunch break **13.30** Lectures and exercises **17.30** We will walk along the Kromme Rijn river to De Moestuin, Laan van Maarschalkerweerd 2, for a barbecue

Tuesday 13 August	**Dr. Michał Wrochna: Black hole spacetimes** This lecture introduces elements of Lorentzian geometry in the context of General Relativity, with a particular focus on causality and examples of black hole spacetimes. We start by discussing notions from Special Relativity from the point of view of geometry, in particular time-like, space-like and null vectors. After a crash course on vector fields, we motivate the definition of Lorentzian metric and discuss curves and causality in General Relativity. An example discussed at length is the Schwarzschild metric: what does it mean that it models a black hole? Next, the notion of Killing vector field is introduced, followed by a discussion of various scenarios on black hole spacetimes. If time permits, the notion of globally hyperbolic spacetime will be motivated and introduced, hand-in-hand with mathematical tools for its study. **Prerequisites.** Standard bachelor's course on multivariable calculus. Differential geometry (differential manifolds or submanifolds, tangent bundle, cotangent bundle, vector fields) is helpful, but not mandatory. **References.** Eric Gourgoulhon, "Geometry and physics of black holes" (open access lecture notes), selected chapters. **09.00** Lectures and exercises **12.30** Lunch break **13.30-17.00** Lectures and exercises

Wednesday 14 August	**Prof. Dr. Ieke Moerdijk and Dr. Tobias Lenz: Representations of finite groups** The representation theory of finite groups studies the ways a finite group can act on a vector space, or equivalently, the ways in which a finite group can be mapped into a matrix group. The result is a beautiful mathematical theory, reducing the problem completely to understanding certain ("conjugate-invariant") functions from the group into the complex numbers, the so-called characters of the group. We will explain some first steps of this theory, study a couple of examples, and explain how representations of different groups are related. **Expected background.** Linear algebra and basic group theory. **09.30-10.15** Lecture (theory) **10.30-11.15** Lecture (theory) **11.30-12.30** Examples/exercises **12.30-13.30** Lunch break **13.30-14.30** Lecture (theory) **15.00-15.45** Lecture (more examples) **16.00-17.00** Exercises and discussion

Thursday 15 August	**Dr. Soumya Sankar: Newton polygons** How do you decide whether a polynomial with integer coefficients has any proper factors? How do you decide if such a polynomial has a rational zero? And if you do know that it has one, how do you find it? Such questions are (provably) hard, and often the best one can do is provide partial answers to them. In this course, we will learn about a powerful tool that helps us study roots of polynomial equations: Newton polygons. Dating back to a letter from Newton to Oldenburg in the 17th century, the study of these polygons has expanded greatly and has found a wide variety of applications in modern mathematics. In this course, we will learn about the construction of Newton polygons and their application to polynomial factorization, finding roots of polynomials and beyond. **Expected background.** Basic abstract algebra (Groups and rings. It would be helpful to know what a field is, but it is not necessary). Primes and divisibility. **09.00** Lecture 1 **10.00** Exercise class 1 and discussion **11.30** Lecture 2 **12.30** Lunch break **13.30** Lecture 3 **14.30-17.00** Exercise class 2 and discussion

Friday 16 August	**Prof. Dr. Jan Hogendijk and Dr. Viktor Blåsjö: History of geometry** The historical evolution of geometry provides a perspective on the subject that is not only mathematically but also philosophically, culturally, and pedagogically enriching. We begin by exploring Euclid's Elements through an interactive visual edition that we have developed. This classic text is intrinsically rewarding for its mathematical depth and beauty, and furthermore gives us occasion to reflect on timeless themes such as the role in mathematics of formalism, intuition, constructions, axiomatisation, and diagrammatic reasoning. We also briefly consider the wide-ranging cultural footprints of Euclid's text, which run across millennia of intellectual history. In the afternoon we will discuss the historical development of the objects that were studied in geometry. In Greek antiquity and the middle ages, geometry was limited to the study of straight lines, circles, conic sections, polyhedra, the sphere, cylinder, cone, and a few other curves and solids, which were all in some sense "constructible". In 1637 Descartes made a connection between geometry and algebra, and as a result algebraic curves were added to geometry, and their "construction" became less and less important over the years. Starting in the second half of the mid-nineteenth century, geometry was no longer limited to three dimensions and pre-existing objects such as the "plane" and "space". Mathematics gradually developed to the study of arbitrary structures invented by the human mind. **09.00** Lectures and exercises **12.30** Lunch break **13.30-17.00** Lectures and exercises

Saturday/Sunday 17 and 18 August	For the social programme organised by UU for all summer school students, see: https://utrechtsummerschool.nl/.

<table>
<tr><td>Monday 19 August</td><td>Prof. Dr. Frans Oort: Elliptic Curves

In algebraic geometry we consider geometric structures given as solutions of polynomial equations. Elliptic curves are non-singular plane cubic curves with a marked rational point. As a fascinating property we see that the set of points on an elliptic curve canonically has the structure of an abelian group. This mixture of geometry and algebra provides us with interesting applications, especially in geometry, and in arithmetic. We give a survey of definitions and properties of elliptic curves.

Among the wide variety of applications, we have chosen one topic. In 1822 Poncelet published a proof of the closure theorem: the existence of one polygon with vertices on one conic and sides tangent to another conic proves there are infinitely many such polygons. We present a proof given by Jacobi in 1828 in modern terminology.

This course will give many exercises to the students. We discuss historical aspects of geometry in the 18th and the 19th century.

09.00 Lectures and exercises
12.30 Lunch break
13.30-17.00 Lectures and exercises</td></tr>
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<tr><td>Tuesday 20 August</td><td>Dr. Niall Taggart and Dr. Guy Boyde: A topologist’s time machine

Topology is the study of topological spaces up to continuous deformation: topologists declare two “shapes” (or spaces) to be “the same” if one can be stretched or squeezed into the “shape” of the other. To classify spaces up to this notion of “sameness” topologists have developed a wide range of algebraic tools to distinguish spaces based on their “shape”.

The starting point was the work of Euler on distinguishing “shapes” based on a simple formula involving the number of vertices, edges and faces. This mini-course will act as a mathematical time machine, taking us through the advances of algebraic topology starting from Euler and moving toward more modern approaches using more sophisticated algebra (groups, rings, etc).

Prerequisites. Point set topology and basic linear algebra.

09.00 Lectures and exercises
12.30 Lunch break
13.30 Lectures and exercises
17.00 We will walk along the Kromme Rijn river to Theehuis Rhijnauwen, Rhijnauwenselaan 16 in Bunnik, for a pancake</td></tr>
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Wednesday 21 August	**Dr. Álvaro del Pino Gómez and Anna Fokma MSc: A gentle introduction to Morse theory** A manifold is a (Hausdorff and second countable) topological space that locally looks like standard Euclidean space. Some familiar examples may be the circle (which has dimension 1) or the surfaces (like the sphere or the torus, which have dimension 2). In a (smooth) manifold we can do calculus/analysis much like we do it in $\mathbb{R}^n$. In particular, we can speak of a function being differentiable (to a given order) and we can study its critical points. The question at the heart of Morse theory is the following: "what can a function tell us about the topology of the manifold?". During the course we will explore this question in the particular case of surfaces. In particular, we will construct an invariant of the manifold called Morse homology. **Prerequisites.** We assume that the student has taken a basic Topology course and the first year Calculus courses. A knowledge of manifold theory, simplicial/singular homology, or de Rham cohomology is potentially useful, but not necessary. **References.** Lecture notes will be provided. Moreover, a standard reference for the topic (including a nice first chapter dedicated solely to surfaces) is the book: Y. Matsumoto. An Introduction to Morse Theory. Translations of Mathematical Monographs. Iwanami Series in Modern Mathematics vol. 208 (2002). **09.00** Lectures and exercises **12.30** Lunch break **13.30-17.00** Lectures and exercises

Thursday 22 August	**Dr. Remy van Dobben de Bruyn: Representations of matrix groups** The group of $n \times n$ invertible matrices contains many interesting subgroups, such as the diagonal matrices, the upper triangular matrices, or the matrices of determinant 1. We will study some more examples, and use tools from linear algebra to construct homomorphisms between them. All the representations obtained in this way are algebraic: their matrix entries are given by polynomial maps. We will see examples of the interplay between properties of (the elements of) the group and properties of its algebraic representations. **Prerequisites.** Linear algebra, analysis in several variables. Some exposure to multivariate polynomials (for instance in ring theory) is useful, but not required. **09.00** Lectures and exercises **12.30** Lunch break **13.30-17.00** Lectures and exercises

Friday 23 August	**Ryan Quinn MSc: Topological quantum field theories and cobordisms** Connections between topology and physics provide deep insights into both subjects. A striking example of this is the relation between cobordisms of manifolds, and topological quantum field theories. In this mini-course we will introduce cobordisms and topological quantum field theories, and explore the relation between them in the 1 and 2-dimensional cases. The main goal is to understand how to make this connection precise in the 1 and 2-dimensional cases, and obtain an appreciation for this striking phenomenon. In these instances, everything can be described explicitly, as such this mini-course will focus on examples. **Prerequisites.** Familiarity with tensor products of vector spaces will be helpful. Familiarity with manifolds would be helpful, but not essential, as we will focus explicitly on 1 and 2-dimensional cases. Ideas from the algebraic topology mini-course will help with motivation. **09.30** Lectures and exercises **12.30** Lunch